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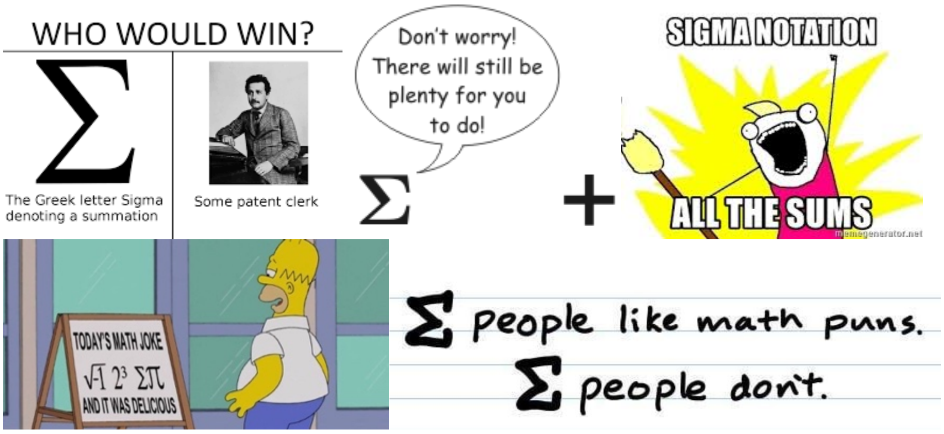
Sequences & Series: Sigma Notation Solutions



This sheet assumes that you already know how to deal with arithmetic and geometric series.

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1)

$$\sum_{r=1}^{r=6}(r+1)$$

In English, this says replace every r starting from 1 in the expression $(r+1)$ and go all the way to 6. We add ($\sum$ means add) all these terms found.

Let's see in detail how this works.

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{r=1}^{r=6}(r+1)$$

To do this we replace $(r+1)$ with the values of r
We know r starts at 1 and ends at 6
This is not many terms so it's easy to write them all out. Normally we just write out the first 3 or 4 terms which is enough to spot the pattern from afterwards

$$(1+1)+(2+1)+(3+1)+(4+1)+(5+1)+(6+1)$$

$$=2+3+4+5+6+7$$

Step 2: Decide what type of series we have
Here we keep adding 1 each time so we have an arithmetic sequence with $a=2$ and $d=1$

Step 3: Find the sum
Way 1: Since we only have a few terms we can find the sum easily: $2+3+4+5+6+7=27$

Way 2: use the s_n formula for an arithmetic series with $a=2$ and $d=1$
we have 6 terms
$S_6=\frac{6}{2}[2(2)+(6-1)(1)]=27$

2)

i.

$$\sum_{k=1}^{k=6}(2k+1)$$

In English, this says replace every k starting from 1 in the expression $(2k+1)$ and go all the way to 6. We add ($\sum$ means add) all these terms found.

Let's see in detail how this works.

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Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{k=1}^{k=6}(2k+1)$$

To do this we replace $(2k+1)$ with the values of k
We know k starts at 1 and ends at 6
This is not many terms so it's easy to write them all out. Normally we just write out the first 3 or 4 terms which is enough to spot the pattern from afterwards

$$(2(1)+1)+(2(2)+1)+(2(3)+1)+(2(4)+1)+(2(5)+1)+(2(6)+1)$$

$$= 3+5+7+9+11+13$$

Step 2: Decide what type of series we have
Here we keep adding 2 each time so we have an arithmetic sequence with $a=3$ and $d=2$

Step 3: Find the sum
Way 1: Since we only have a few terms we can find the sum easily: $3+5+7+9+11+13=48$

Way 2: use the s_n formula for an arithmetic series with $a=3$ and $d=2$
we have 6 terms
$S_6=\frac{6}{2}[3(2)+(6-1)(2)]=48$

ii.

$$\sum_{k=3}^{k=7}(k^2)$$

In English, this says replace every k starting from 3 in the expression (k^2) and go all the way to 7. We add ($\sum$ means add) all these terms found.

Let's see in detail how this works.

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{k=3}^{k=7}(k^2)$$

To do this we replace (k^2) with the values of k
We know k starts at 3 and ends at 7
This is not many terms so it's easy to write them all out. Normally we just write out the first 3 or 4 terms which is enough to spot the pattern from afterwards

$$(3^2)+(4^2)+(5^2)+(6^2)+(7^2)$$

$$=9+16+25+36+49$$

Step 2: Decide what type of series we have
: There is no real pattern to it, so adding the terms

Step 3: Find the sum

We only have a few terms though, so we can find the sum easily:

$$9+16+25+36+49$$
$$=135$$

Note: This series is neither arithmetic nor geometric, so we can't use the s_n formula.

iii.

$$\sum_{k=4}^{k=9}k(2^{2k-1})$$

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In English, this says replace every k starting from 4 in the expression $(k(2^{2k-1})$ and go all the way to 9. We add ($\sum$ means add) all these terms found.

Let's see in detail how this works.

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{k=4}^{k=9} k(2^{2k-1})$$

To do this we replace $k(2^{2k-1})$ with the values of k

We know k starts at 4 and ends at 9

This is not many terms so it's easy to write them all out. Normally we just write out the first 3 or 4 terms which is enough to spot the pattern from afterwards

$$4(2^{2(4)-1}) + 5(2^{2(5)-1}) + 6(2^{2(6)-1}) + 7(2^{2(7)-1}) + 8(2^{2(8)-1}) + 9(2^{2(9)-1})$$

$$= 512 + 2560 + 12288 + 57344 + 262144 + 1179648$$

Step 2: Decide what type of series we have

This is a geometric series, but also with a k term in front, so adding all of those terms

Step 3: Find the sum

Since we only have a few terms we can find the sum easily:

$$512 + 2560 + 12288 + 57344 + 262144 + 1179648$$

$$= 1514496$$

Note: This series is neither arithmetic nor geometric, so we can't use the s_n formula.

iv.

$$\sum_{k=100}^{k=100} k(2^{2k-1})$$

In English, this says replace every k starting from 100 in the expression $(3k-7)$ and also stop at 100. We add ($\sum$ means add) all these terms found.

Let's see in detail how this works.

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{k=100}^{k=100} (3r-7)$$

To do this we replace $(3k-7)$ with the values of k

We know k starts at 100 and ends at 100

This is only 1 term

$$3(100) - 7$$

$$= 293$$

Step 2: Decide what type of series we have

Here we have a single term, so we just have that as the answer

Step 3: Find the sum

Since we only have 1 term it is just 293

3)

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$$\sum_{r=1}^{r=5}(3r)$$

In English, this says replace every r starting from 1 in the expression given $(3r)$ and go all the way to 5.
We add ($\sum$ means add) all these terms found.

Let's see in detail how this works.

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{r=1}^{r=5}(3r)$$

To do this we replace $(3r)$ with the values of r
We know r starts at 1 and ends at 5
This is not many terms so it's easy to write them all out. Normally we just write out the first 3 or 4 terms which is enough to spot the pattern from afterwards

$$(3(1)) + (3(2)) + (3(3)) + (3(4)) + (3(5))$$

$$= 3 + 6 + 9 + 12 + 15$$

Step 2: Decide what type of series we have
Here we keep adding 1 each time so we have an arithmetic sequence with $a = 3$ and $d = 3$

Step 3: Find the sum
Way 1: Since we only have a few terms we can find the sum easily: $3 + 6 + 9 + 12 + 15$ = 45

Way 2: use the s_n formula for an arithmetic series with $a = 3$ and $d = 3$
we have 5 terms
$S_5 = \frac{5}{2}[3(2) + (5-1)(3)] = 45$

4)

$$\sum_{r=0}^{r=5}(r+1)$$

In English, this says replace every r starting from 0 in the expression given $(r+1)$ and go all the way to 5.
We add ($\sum$ means add) all these terms found.

Let's see in detail how this works.

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{r=0}^{r=5}(r+1)$$

To do this we replace $(r+1)$ with the values of r
We know r starts at 0 and ends at 5
This is not many terms so it's easy to write them all out. Normally we just write out the first 3 or 4 terms which is enough to spot the pattern from afterwards

$$(0+1) + (1+1) + (2+1) + (3+1) + (4+1) + (5+1)$$

$$= 1 + 2 + 3 + 4 + 5 + 6$$

Step 2: Decide what type of series we have
Here we keep adding 1 each time so we have an arithmetic sequence with $a = 1$ and $d = 1$

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Step 3: Find the sum
Way 1: Since we only have a few terms we can find the sum easily: $1+2+3+4+5+6=21$

Way 2: use the s_n formula for an arithmetic series with $a=1$ and $d=1$
we have 6 terms
$S_6=\frac{6}{2}[2(1)+(6-1)(1)]=21$

5)

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{r=1}^{r=20}(4r+1)$$

To do this we replace $(4r+1)$ with the values of r
We know r starts at 1 and ends at 20.
This is quite a lot of terms so let's just write few terms out which will be enough to spot the pattern from afterwards

$$(4(1)+1)+(4(2)+1)+(4(3)+1)+(4(4)+1)+(4(5)+1)+(4(6)+1)+\cdots+(4(20)+1)$$

$$=5+9+13+17+\cdots+81$$

Step 2: Decide what type of series we have
Here we keep adding 4 each time so we have an arithmetic sequence with $a=5$ and $d=4$

Step 3: Find the sum
use the s_n formula for an arithmetic series with $a=5$ and $d=4$
we have 20 terms
$S_{20}=\frac{20}{2}[2(5)+(20-1)(4)]=860$

6)

$$\sum_{r=1}^{r=20}(5r-2)$$

In English, this says replace every r starting from 1 in the expression given $(5r-2)$ and go all the way to 20. We add ($\sum$ means add) all these terms found.

Let's see in detail how this works.

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{r=1}^{r=20}(5r-2)$$

To do this we replace $(5r-2)$ with the values of r
We know r starts at 1 and ends at 20
This is not many terms so it's easy to write them all out. Normally we just write out the first 3 or 4 terms which is enough to spot the pattern from afterwards

$$(5(1)-2)+(5(2)-2)+(5(3)-2)+\cdots$$

$$=3+8+13+\cdots$$

Step 2: Decide what type of series we have
Here we keep adding 1 each time so we have an arithmetic sequence with $a=3$ and $d=5$

Step 3: Find the sum

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Way 1: use the s_n formula for an arithmetic series with $a = 3$ and $d = 5$
we have 20 terms
$S_6 = \frac{20}{2}[2(3) + (20 - 1)(5)] = 1010$

7)

$$\sum_{r=1}^{r=42}(5r+3)$$

In English, this says replace every r starting from 1 in the expression given $(5r + 3)$ and go all the way to 42. We add ($\sum$ means add) all these terms found.

Let's see in detail how this works.

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{r=1}^{r=42}(5r+3)$$

To do this we replace $(5r + 3)$ with the values of r
We know r starts at 1 and ends at 42
This is not many terms so it's easy to write them all out. Normally we just write out the first 3 or 4 terms which is enough to spot the pattern from afterwards

$$(5(1) + 3) + (5(2) + 3) + (5(3) + 3) + \cdots$$

$$= 8 + 13 + 18 + \cdots$$

Step 2: Decide what type of series we have
Here we keep adding 1 each time so we have an arithmetic sequence with $a = 8$ and $d = 5$

Step 3: Find the sum

Way 1: use the s_n formula for an arithmetic series with $a = 8$ and $d = 5$
we have 42 terms
$S_{42} = \frac{42}{2}[2(8) + (20 - 1)(5)] = 2331$

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2 Silver

8)

i.

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{n=1}^{n=20} 3n$$

To do this we replace $3n$ with the values of n
We know n starts at 1and ends at 20.
This is quite a lot of terms so let's just write few terms out which will be enough to spot the pattern from afterwards

$$3(1)+3(2)+3(3)+3(4)+\cdots+60$$

$$=3+6+9+12+\cdots 60$$

Step 2: Decide what type of series we have
Here we keep adding 3 each time so we have an arithmetic sequence with $a=3$ and $d=3$

Step 3: Find the sum
use the s_n formula for an arithmetic series with $a=3$ and $d=3$
we have 20 terms
$S_{20}=\frac{20}{2}[2(3)+(20-1)(3)]=630$

ii.

$$\sum_{n=21}^{n=100} 3n$$

Notice how here we don't start from 1. There are 2 ways to deal with this. Force the sum to start from 1 or deal with it as it currently is.

Way 1: Start the series from 21

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{n=21}^{n=100} 3n$$

To do this we replace $3n$ with the values of n
We know n starts at 21 and ends at 200

Way 2: Force the series to start from 1

Using the fact that $\sum_m^n \ldots = \sum_1^n \ldots - \sum_1^{m-1} \ldots$

$$\sum_{r=21}^{100} = \sum_{r=1}^{100} - \sum_{r=1}^{20}$$

$$=S_{100}-S_{20}$$

$$=\frac{100}{2}[2(63)+(80-1)(3)]-\frac{20}{2}[2(63)+(20-1)(3)]$$

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$3(21)+3(22)+3(23)+3(24)+\cdots+3(100)$ $= 63+66+69+12+\cdots+300$ **Step 2:** We know this is arithmetic already from part i. **Step 3:** Find the sum Now we need to be careful since it is not as obvious how many terms we have since the sum doesn't start from 1. There are 2 ways to find the number of terms. Way 1: $\sum_{r=m}^{n}$ generally has $m-n+1$ terms So here we have $100-21+1=63$ terms Way 2: The last term is 300 We can solve $u_n=300$ This will tell us how many terms we have $u_n=a+(n-1)d$ $63+(n-1)3=300$ $63+3n-3=300$ $3n=240$ $n=80$ use the s_n formula for an arithmetic series with $a=63$ and $d=3$. We have 80 terms $S_{80}=\frac{80}{2}[2(63)+(80-1)(3)]=14{,}520$	$=15{,}150-630$ $=14{,}520$

9)

$$\sum_{r=10}^{r=30}(7+2r)$$

Notice how here we don't start from 1. There are 2 ways to deal with this. Force the sum to start from 1 or deal with it as it currently is.

Way 1: Start the series from 10	**Way 2: Force the series to start from 1** **Using the fact that $\sum_m^n \ldots = \sum_1^n \ldots - \sum_1^{m-1}\ldots$**
Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like. $\sum_{r=10}^{r=30}(7+2r)$ We know r starts at 10 and ends at 30 $(2(10)+7)+(2(11)+7)+\cdots+(2(30)+7)$ $=27+29+\cdots+67$ **Step 2:** This is arithmetic	$\sum_{r=10}^{30} = \sum_{r=1}^{30} - \sum_{r=1}^{9}$ $S_{100}-S_{20}$ $=\frac{30}{2}[2(9)+(30-1)(2)]-\frac{9}{2}[2(9)+(9-1)(2)]$ $=1140-153$ $=987$

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Step 3: Find the sum Now we need to be careful since it is not as obvious how many terms we have since the sum doesn't start from 1. There are 2 ways to find the number of terms. Way 1: $\sum_{r=m}^{n}$ generally has $m - n + 1$ terms So here we have $30 - 10 + 1 = 21$ terms Way 2: The last term is 67 We can solve $u_n = 67$ This will tell us how many terms we have $u_n = a + (n-1)d$ $27 + (n-1)2 = 67$ $27 + 2n - 2 = 67$ $n = 21$ use the s_n formula for an arithmetic series with $a = 27$ and $d = 2$. We have 21 terms $S_{21} = \frac{21}{2}[2(27) + (21-1)(2)] = 987$	

10)

$$\sum_{r=3}^{r=6}(2^r - 1)$$

Notice how here we don't start from 1. There are 2 ways to deal with this. Force the sum to start from 1 or deal with it as it currently is.

Way 1: Start the series from 3	**Way 2: Force the series to start from 1** **Using the fact that $\sum_m^n \ldots = \sum_1^n \ldots - \sum_1^{m-1} \ldots$**
Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like. $\sum_{r=3}^{r=6}(2^r - 1)$ To do this we replace $2^r - 1$ with the values of r We know r starts at 3 and ends at 6 $(2^3 - 1) + (2^4 - 1) + (2^5 - 1) + (2^6 - 1)$ $= 7 + 15 + 31 + 63$ **Step 2:** This is a model which we can do the sum as there aren't many terms **Step 3:** Find the sum This one does not have many terms so we can work it out $7 + 15 + 31 + 63$ $= 116$	$\sum_{r=3}^{6} = \sum_{r=1}^{6} - \sum_{r=1}^{2}$ $S_6 - S_2$ $S_6 = (2^1 - 1) + (2^2 - 1) + .. + (2^6 - 1)$ $S_6 = 120$ $S_2 = (2^2 - 1) + (2^1 - 1)$ $S_2 = 4$ $S_6 - S_2 = 116$

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11)

$$\sum_{r=-1}^{r=4} (1.5^r)$$

Notice how here we don't start from 1. We could force it to start from one, but that does not make much sense here to do so, so we can just keep it as starting from -1

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{r=-1}^{r=4} (1.5^r)$$

To do this we replace 1.5^r with the values of r
We know n starts at -1 and ends at 4

$$1.5^{-1} + 1.5^0 + 1.5^1 \ldots + 1.5^4$$

$$= \frac{2}{3} + 1 + 1.5 + \cdots + \frac{81}{16}$$

Step 2: This is a geometric sequence with
a = $\frac{2}{3}$ r = $\frac{3}{2}$

Way 1: Use a formula

Now we need to be careful since it is not as obvious how many terms we have since the sum doesn't start from 1. There are 2 ways to find the number of terms.

Way 1:	Way 2:
$\sum_{r=m}^{n}$ generally has (n−m)+1 terms when m < 0 So here we have $(4 - -1) + 1 = 6$ terms	The last term is $\frac{81}{16}$ We can solve $u_n = \frac{81}{16}$ This will tell us how many terms we have $u_n = a(r^{n-1})$ $\frac{2}{3}\left(\frac{3}{2}\right)^{n-1} = \frac{81}{16}$ $\left(\frac{3}{2}\right)^{n-1} = \frac{243}{32}$ $n = 6$

Using the sum formula

Way 2: There aren't many terms so we can just add the terms

There are only 6 terms, so add them

$$\frac{2}{3} + 1 + 1.5 + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} = \frac{665}{48}$$

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$S_n = \frac{a(1-r^n)}{1-r}$ $S_6 = \frac{\frac{2}{3}(1-1.5^6)}{1-1.5}$ $S_6 = \frac{665}{48}$	

12)

i.

Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{r=4}^{r=7} 2^r$$

To do this we replace 2^r with the values of r

We know n starts at 4 and ends at 7

$$2^4 + 2^5 + 2^6 + 2^7$$
$$16 + 32 + 64 + 128 = 240$$

ii.

$$\sum_{r=4}^{r=30} 2^r$$

Notice how here we don't start from 1. We have two options with how to deal with this

<table>
<tr><th>Way 1: Start the series from 4</th><th>Way 2: Force the series to start from 1</th></tr>
<tr><td>

Step 1: Write a few terms out i.e. get rid of the sigma notation to see what the sequence looks like.

$$\sum_{r=4}^{r=30} 2^r$$

To do this we replace 2^r with the values of n
We know n starts at 4 and ends at 30

$$2^4 + 2^5 + \cdots + 2^{30}$$

$$= 16 + 32 + 64 + \cdots + 1073741824$$

Step 2: This is a geometric sequence with
a = 16 d = 2

Step 3: Find the sum
Now we need to be careful since it is not as obvious how many terms we have since the sum doesn't start from 1. There are 2 ways to find the number of terms.

<table>
<tr><th>Way 1:</th><th>Way 2:</th></tr>
<tr><td>$\sum_{r=m}^{n}$ generally has (n-m)+1 terms when m < 0</td><td>The last term is 1073741824</td></tr>
</table>

</td><td>

$$\sum_{n=1}^{n=30} 2^r - \sum_{n=1}^{n=3} 2^r = \sum_{n=4}^{n=30} 2^r$$

$$S_{30} - S_3 = \frac{2(1-2^{30})}{1-2} - \frac{2(1-2^3)}{1-2}$$

$$S_{30} = 2147483646 - 14$$

$$= 2147483632$$

</td></tr>
</table>

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So here we have (30-4) $+$ 1 $=$ 27 terms	We can solve $u_n = 1073741824$ This will tell us how many terms we have $u_n = ar^{n-1}$ $16(2)^{n-1} = 1073741824$ $(2)^{n-1} = 67108864$ $n = 27$

Using the sum formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_{27} = \frac{16(1 - 2^{27})}{1 - 2}$$

$$S_{27} = 2147483632$$

iii.
The ratio is 2, for a system to converge: $|r|$<1, this system diverges

13)

i.

To do this we replace $2r - 5$ with the values of n

$$2(1) - 5\,, 2(2) - 5\,, 2(3) - 5$$
$$-3, -1, 1$$

ii.

Using part i, look at term one and two and subtract them:

$$-1 - -3 = 2$$

iii.

Using part i. a$=$ -3
Using part ii. d = 2

Using the formula for sum of an arithmetic series:

$$S_n = \frac{\mathrm{n}}{2}(2(a) + (n - 1)d)$$

Subbing in the values

$$s_n = \frac{\mathrm{n}}{2}(2(-3) + 2(n - 1))$$

$$= \frac{\mathrm{n}}{2}(-6 + 2n - 2)$$

$$= \frac{\mathrm{n}}{2}(2n - 8)$$

$$= n(n - 4)$$

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14)

Start by listing some of the terms

$$\sum_{r=1}^{n} r$$

$$1+2+3+4+\cdots$$

Now we can establish the constants we need:
first term= $a=1$
common difference = d = 1
number of terms = n

Using the formula for sum of an arithmetic series:

$$S_n = \frac{n}{2}(2(a)+(n-1)d)$$

Subbing in the values

$$S_n = \frac{n}{2}(2(1)+1(n-1))$$

$$=\frac{n}{2}(2+n-1)$$

$$=\frac{n}{2}(n+1)$$

15)

$$\sum_{r=1}^{6} 10\times\left(\frac{2}{3}\right)^{r-1}$$

Way 1: Leave the constant inside

$$\sum_{r=1}^{6} 10\times\left(\frac{2}{3}\right)^{r-1}$$

Start by listing some of the terms of the series

$$=10\left(\frac{2}{3}\right)^0+10\left(\frac{2}{3}\right)^1+10\left(\frac{2}{3}\right)^2+\cdots$$

$$=10+\frac{20}{3}+\frac{40}{9}+\cdots$$

Now we can establish the constants we need:
first term= $a=10$
common ratio= $r=\frac{2}{3}$
number of terms = 6

Using the formula for sum of a geometric series:

$$=\frac{a(1-r^n)}{(1-r)}$$

Subbing in the values:

Way 2: Take out the constant

The sum only depends on r so we can take the 10 out (like with integration)

$$10\sum_{r=1}^{6}\left(\frac{2}{3}\right)^{r-1}$$

Ignore the 10.
Start by listing some of the terms of the series

$$=\left(\frac{2}{3}\right)^0+\left(\frac{2}{3}\right)^1+\left(\frac{2}{3}\right)^2+\cdots$$

$$=1+\frac{2}{3}+\frac{4}{9}+\cdots$$

Now we can establish the constants we need:
first term= $a=1$
common ratio= $r=\frac{2}{3}$
number of terms = 6

Using the formula for sum of geometric series:

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$$=\frac{10\left(1-\left(\frac{2}{3}\right)^6\right)}{1-\frac{2}{3}}$$ Simplifying the terms $$=\frac{\frac{6650}{729}}{\frac{1}{3}}$$ $$=\frac{6650}{243}$$	$$=\frac{a(1-r^n)}{(1-r)}$$ Subbing in the values: $$=\frac{1\left(1-\left(\frac{2}{3}\right)^6\right)}{1-\frac{2}{3}}$$ Simplifying the terms $$=\frac{\frac{665}{729}}{\frac{1}{3}}$$ $$=\frac{665}{243}$$ Remembering the factor 10 we took out, we need to time that back in $$=\frac{665}{243}\times 10$$ $$=\frac{6650}{243}$$

16)

$$\sum_{r=1}^{\infty} 10\times\left(\frac{2}{3}\right)^{r-1}$$

Way 1: Leave the constant inside	**Way 2: Take out the constant**
$$\sum_{r=1}^{\infty} 10\times\left(\frac{2}{3}\right)^{r-1}$$ Start by listing some of the terms of the series $$=10\left(\frac{2}{3}\right)^0+10\left(\frac{2}{3}\right)^1+10\left(\frac{2}{3}\right)^2+\cdots$$ $$=10+\frac{20}{3}+\frac{40}{9}+\cdots$$ Now we can establish the constants we need: first term= $a=10$ common ratio= $r=\frac{2}{3}$ we have an infinite number of term, so we want to use the sum to infinity Using the formula for sum of sum to infinity: $$=\frac{a}{1-r}$$	**The sum only depends on r so we can take the 10 out (like with integration)** $$10\sum_{r=1}^{\infty}\left(\frac{2}{3}\right)^{r-1}$$ Ignore the 10. Start by listing some of the terms of the series $$=\left(\frac{2}{3}\right)^0+\left(\frac{2}{3}\right)^1+\left(\frac{2}{3}\right)^2+\cdots$$ $$=1+\frac{2}{3}+\frac{4}{9}+\cdots$$ Now we can establish the constants we need: first term= $a=1$ common ratio= $r=\frac{2}{3}$ we have an infinite number of term, so we want to use the sum to infinity

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Subbing in the values: $$=\frac{10}{1-\frac{2}{3}}$$ Simplifying the terms $$=\frac{10}{\frac{1}{3}}$$ $$=30$$	Using the formula for sum of sum to infinity: $$=\frac{a}{1-r}$$ Subbing in the values: $$=\frac{1}{1-\frac{2}{3}}$$ Simplifying the terms $$=\frac{1}{\frac{1}{3}}$$ $$=3$$ Remembering the factor 10 we took out, we need to time that back in $$=3\times 10$$ $$=30$$

17)

$$\sum_{r=7}^{\infty} 10\times\left(\frac{2}{3}\right)^{r-1}$$

Way 1: Leave the constant inside	**Way 2: Take out the constant**
$$\sum_{r=7}^{\infty} 10\times\left(\frac{2}{3}\right)^{r-1}$$ Start by listing some of the terms of the series $$=10\left(\frac{2}{3}\right)^{7-1}+10\left(\frac{2}{3}\right)^{8-1}+10\left(\frac{2}{3}\right)^{9-1}$$ $$=\frac{640}{729}+\frac{1280}{2187}+\frac{2560}{65561}$$ Now we can establish the constants we need: first term= $a=\frac{640}{729}$ common ratio= $r=\frac{2}{3}$ we have an infinite number of terms, so we want to use the sum to infinity Using the formula for sum of sum to infinity: $$=\frac{a}{1-r}$$ Subbing in the values: $$=\frac{\frac{640}{729}}{1-\frac{2}{3}}$$	**The sum only depends on r so we can take the 10 out (like with integration)** $$10\sum_{r=7}^{\infty}\left(\frac{2}{3}\right)^{r-1}$$ Ignore the 10. Start by listing some of the terms of the series $$=\left(\frac{2}{3}\right)^{7-1}+\left(\frac{2}{3}\right)^{8-1}+\left(\frac{2}{3}\right)^{9-1}$$ $$=\frac{64}{729}+\frac{128}{2187}+\frac{256}{65561}$$ Now we can establish the constants we need: first term= $a=\frac{64}{729}$ common ratio= $r=\frac{2}{3}$ we have an infinite number of terms, so we want to use the sum to infinity Using the formula for sum of sum to infinity: $$=\frac{a}{1-r}$$ Subbing in the values:

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Simplifying the terms $$=\frac{\frac{640}{729}}{\frac{1}{3}}$$ $$=\frac{640}{243}$$	$$=\frac{\frac{64}{729}}{1-\frac{2}{3}}$$ Simplifying the terms $$=\frac{\frac{64}{729}}{\frac{1}{3}}$$ $$=\frac{64}{243}$$ Remembering the factor 10 we took out, we need to time that back in $$=\frac{64}{243}\times 10$$ $$=\frac{640}{243}$$

18)

Start by taking out the constant 20 (we can take this out since the sum depends on r only. We don't have to though)

$$20\sum_{r=4}^{\infty}\left(\frac{1}{2}\right)^r$$

Start by listing some of the terms (remember the sequence starts from 4)

$$=\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^5+\left(\frac{1}{2}\right)^6$$

$$=\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$$

Now we can establish the constants we need:
first term= $a=\frac{1}{16}$
common ratio= $r=\frac{1}{2}$
Using the formula for sum to infinity:

$$=\frac{a}{1-r}$$

Subbing in the values:

$$=\frac{\frac{1}{16}}{1-\frac{1}{2}}$$

Simplifying the terms

$$=\frac{\frac{1}{16}}{\frac{1}{2}}$$

$$=\frac{1}{8}$$

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Remembering the factor 20 we took out, we need to time that back in

$$= \frac{1}{8} \times 20$$

$$= 2.5$$

19)

Start by listing some of the terms

$$\sum_{r=1}^{n} (2r - 3)$$

$$= (2(1) - 3) + (2(2) - 3) + (2(3) - 3) + \cdots$$

$$= -1 + 1 + 3 + \cdots$$

Now we can establish the constants we need:
first term= $a = -1$
common difference =d = 2
number of terms = n

Using the formula for sum of arithmetic series:

$$S_n = \frac{\text{n}}{2}(2(a) + (n-1)d)$$

Subbing in the values

$$S_n = \frac{\text{n}}{2}(2(-1) + 2(n-1))$$

$$= \frac{\text{n}}{2}(-2 + 2n - 2)$$

$$= \frac{\text{n}}{2}(2n - 4)$$

$$= n(n-2)$$

ii.

Using the equation we found in part i, and letting n = 50

$$S_n = 50(50 - 2) = 2400$$

iii.
Given $S_n = 575$

We just need to set the equation equal to the value found

$$n(n-2) = 575$$

$$n^2 - 2n = 575$$

$$n^2 - 2n - 575 = 1$$

$$0 = n^2 - 2n - 575 = 0$$

$$(n - 25)(n + 23) = 0$$

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$$n = 25, n = -23$$

$n = 25$ (since n can't be negative)

20)

Start by listing some of the terms

$$\sum_{r=1}^{n} 3 \times 5^{r-1}$$

$$= (3 \times 5^{1-1}) + (3 \times 5^{2-1}) + (3 \times 5^{3-1})$$

$$= 3 + 15 + 75 + \cdots$$

Now we can establish the constants we need:
first term= $a = 2$
common ratio= $r = 5$
number of terms = m

Using the formula for sum of an arithmetic series:

$$= \frac{a(1-r^m)}{1-r}$$

Subbing in the values:

$$= \frac{3(1-5^m)}{1-5}$$

Told $S_m = 7324218$

$$\frac{3(1-5^m)}{1-5} = 7324218$$

Getting rid of the fractions

$$3(1-5^m) = -29296872$$

Dividing by 3

$$1 - 5^m = -97765624$$

Solving for n

$$m = 10$$

21)

Start by listing some of the terms

$$\sum_{r=1}^{n} (5r+3)$$

$$= (5(1)+3) + (5(2)+3) + (5(3)+3)$$

$$= 8 + 13 + 18 + \cdots$$

Now we can establish the constants we need:
first term= $a = 8$
common difference= $d = 5$
number of terms = n

Using the formula for sum of an arithmetic series:

$$= \frac{n}{2}(2a + d(n-1))$$

Form the inequality:

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$$\frac{n}{2}\left(2a+d(n-1)\right)>1000$$

Subbing in the values

$$\frac{n}{2}\left(16+5(n-1)\right)>1000$$

$$\frac{n}{2}(5n+11)>1000$$

Expanding the brackets and removing the fractions

$$5n^2+11n>2000$$

Making this into a quadratic

$$5n^2+11n-2000>0$$

Making use of the quadratic formula

$$\mathrm{n}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Remember to graph the left hand side or use a number line to solve the inequality

$$n>18.9, n<-21.13$$

Using the decimals, and knowing that we cannot have a negative n

$$n>19$$

So n = 19

22)

Start by listing some of the terms

$$\sum_{r=1}^{n}(100-3r)$$

$$=\left(100-3(1)\right)+\left(100-3(2)\right)+\left(100-3(3)\right)+\cdots$$

$$=97+94+91+\cdots$$

Now we can establish the constants we need:
first term= $a=97$
common difference= $d=-3$
number of terms = n

Using the formula for sum of an arithmetic series:

$$=\frac{n}{2}(2a+d(n-1))$$

Form the inequality:

$$\frac{n}{2}\left(2a+d(n-1)\right)<0$$

Subbing in the values

$$\frac{n}{2}\left(194-3(n-1)\right)<0$$

$$\frac{n}{2}(194-3n+3)<0$$

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$$\frac{n}{2}(197 - 3n) < 0$$

Now looking at the second bracket, set it equal to 0

$$n(197 - 3n) < 0$$

Remember to graph the left-hand side or use a number line to solve the inequality

$$n < 0, n > \frac{197}{3} = 65.6666666$$

$$n = 66$$

23)

Start by listing some of the terms

$$\sum_{r=1}^{n}(100 - 4r)$$

$$= (100 - 4(1)) + (100 - 4(2)) + (100 - 4(3)) + \cdots$$

$$= 96 + 92 + 88 + \cdots$$

Now we can establish the constants we need:
first term= $a = 96$
common difference= $d = -4$
number of terms = n

Using the formula for sum of arithmetic:

$$= \frac{n}{2}(2a + d(n - 1))$$

$$\frac{n}{2}(2a + d(n - 1)) = 0$$

Subbing in the values

$$\frac{n}{2}(192 - 4(n - 1)) = 0$$

$$\frac{n}{2}(192 - 4n + 4) = 0$$

$$\frac{n}{2}(196 - 4n) = 0$$

Now looking at the second bracket, set it equal to 0

$$196 - 4n = 0$$

$$196 = 4n$$

$$50 = n$$

$$n = 50$$

24)

i.

We have been given the formula:

$$\sum_{r=1}^{n} a_r = 12 + 4n^2$$

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$\sum_{r=1}^{5} a_r$ means we replace r with 5

$$12 + 4(5)^2$$
$$= 112$$

For a_6, we need to work out the sum of the first 6 terms, and then subtract the sum of the first 5 terms to give the 6th term:

$$\sum_{r=1}^{6} a_r = 12 + 4(6)^2$$
$$= 156$$

$$a_6 = 156 - 112$$
$$a_6 = 44$$

ii.

For the second part, start by listing some terms

$$\sum_{r=0}^{\infty} \frac{a}{4^r} = \frac{a}{4^0} + \frac{a}{4^1} + \frac{a}{4^2} + \frac{a}{4^3} = a + \frac{a}{4} + \frac{a}{16} + \frac{a}{64} + \cdots$$

Now we can establish the constants we need
Notice how this is a geometric sequence, with:
first term $= \frac{a}{4}$
common ratio $= \frac{1}{4}$

$\sum_{r=0}^{\infty} \frac{a}{4^r} = S_\infty = 16$

$$\text{Sum to infinity} = \frac{a}{1-r}$$

We have

$$\frac{a}{1 - \frac{1}{4}} = 16$$

$$\frac{a}{\frac{3}{4}} = 16$$

$$a = 12$$

25)

Writing in the correct form

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} (2n + 7)$$

Start by listing some of the terms

$$= (2(1) + 7) + (2(2) + 7) + (2(3) + 7) + \cdots$$

$$= 9 + 11 + 13 + \cdots$$

Now we can establish the constants we need:
first term $= a = 9$
common difference $= d = 2$
number of terms = N

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Using the formula for sum of arithmetic series:

$$= \frac{N}{2}(2a + d(N-1))$$

We are given that this sum $= 2100$

$$\frac{N}{2}(2a + d(N-1)) = 2100$$

Subbing in the values

$$\frac{N}{2}(18 + 2(N-1)) = 2100$$

$$\frac{N}{2}(18 + 2N - 2) = 2100$$
$$\frac{N}{2}(16 + 2N) = 2100$$
$$N(8 + N) = 2100$$

Now looking at the second bracket, set it equal to 0

$$N^2 + 8N = 2100$$

$$N^2 + 8N - 2100 = 0$$

Using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$N = 42, -50$$

But the value cannot be negative

$$N = 42$$

26)

Hint: $a = \frac{9}{16}, r = -\frac{3}{4}, find\ S_\infty$

$$\sum_{r=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)$$

Start by listing some of the terms (remember the sequence starts from 2)

$$= \left(\frac{3}{4}\right)^2 \cos(180 \times 2) + \left(\frac{3}{4}\right)^3 \cos(180 \times 3) + \left(\frac{3}{4}\right)^4 \cos(180 \times 4) + \cdots$$

$$= \frac{9}{16} + -\frac{27}{64} + \frac{81}{256}$$

Now we can establish the constants we need:
first term$= a = \frac{9}{16}$
common ratio$= r = -\frac{3}{4}$
we have an infinite number of terms so want to use the sum to infinity formula

Using the formula for sum to infinity:

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$$= \frac{\frac{9}{16}}{1-\left(-\frac{3}{4}\right)}$$

Subbing in the values:

$$= \frac{\frac{9}{16}}{\frac{7}{4}}$$

Simplifying the terms

$$= \frac{9}{28}$$

27)

Start by taking out the constant k (we can take this out since the sum depends on r only. We don't have to though)

$$k\sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^r$$

Start by listing some of the terms

$$= \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3$$

$$= \frac{1}{3} + \frac{1}{9} + \frac{1}{27}$$

Now we can establish the constants we need:
first term= $a = \frac{1}{3}$
common ratio= $r = \frac{1}{3}$
we have an infinite number of terms so want to use the sum to infinity formula

Using the formula for sum to infinity:

$$\frac{a}{1-r}$$

Subbing in the values:

$$= \frac{\frac{1}{3}}{1-\frac{1}{3}}$$

Simplifying the terms

$$= \frac{\frac{1}{3}}{\frac{2}{3}}$$

$$= \frac{1}{2}$$

Put the constant back
Writing this in the form of an equation:

$$\frac{1}{2}k = 7$$

$$k = 14$$

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28)

To write in sigma notation, we need to first establish an nth term:

first term= $a = 4$
common difference $= d = 3$

We need to work out the number of terms:
We have been given the last term, 31, so subbing into the formula

$$= a + d(n-1)$$

We can say

$$31 = 4 + 3(n-1)$$

$$n = 10$$

Now working out the nth term

$$u_n = 4 + 3(n-1)$$
$$= 4 + 3n - 3$$
$$= 3n + 1$$

So now we have an nth term, and we have the number of items:

$$\sum_{r=1}^{10} 3r + 1$$

29)

To write in sigma notation, we need to first establish an nth term:

first term= $a = 40$
common difference $= d = -4$

We need to work out the number of terms:
We have been given the last term, 10, so subbing into the formula

$$= a + d(n-1)$$

We can say

$$0 = 40 - 4(n-1)$$

$$n = 11$$

Now working out the nth term

$$u_n = 40 - 4(n-1)$$
$$= 40 - 4n + 4$$
$$= -4n + 44$$

So now we have an nth term, and we have the number of items:

$$\sum_{r=1}^{11}(-4r + 44)$$

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30)

We notice that this is a geometric series, and nearly all of the information is already given to us:

$$3+3^2+3^3+3^4+\cdots+3^8$$

first term= $a=3$
common ratio $=r=3$
the number of terms = 8

Now working out the nth term

$$u_n=3(3)^{n-1}$$

Using the fact that the formula for an item in a geometric series:

$$\sum_{r=1}^{8}3(3)^{r-1}=\sum_{r=1}^{8}3^r$$

31)

To write in sigma notation, we need to first establish an nth term:

Start by writing out the first few terms:

$$6+12+18\ldots$$

first term= $a=6$
common difference $=d=6$

We need to work out the number of terms:
To work out the last term, start by dividing 100 by 6

$$\frac{100}{6}=16.666$$

Therefore the last integer multiple is 6 \× 16 = 96

This also means that the value of n we go up to is 16

Now working out the nth term

$$\begin{aligned}&=6+6(n-1)\\&=6+6n-6\\&=6n\end{aligned}$$

So now we have an nth term, and we have the number of items:

$$\sum_{r=1}^{16}6n$$

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3 Gold

32)

Start with the information we have been given
The third term is 28
The 6th term is 8
Subbing these into the equation:

$$\textit{third term} = 27 = ar^{3-1} = 27 = ar^2$$

$$\textit{sixth term} = 8 = ar^{6-1} = 8 = ar^5$$

Divide the 2 terms (trick to solve simultaneously:

$$\frac{8}{27} = \frac{ar^5}{ar^2}$$

Simplifying

$$\frac{8}{27} = r^3$$

$$r = \frac{2}{3}$$

Subbing into any of the 2 equations:

$$27 = a\left(\frac{2}{3}\right)^2$$

$$a = \frac{243}{4}$$

We need to find the value of $\sum_{r=6}^{\infty} u_r$

$$u_6 + u_7 + \cdots$$

first term =a= 8 (since the first term is u_6)
common ratio =d= $\frac{2}{3}$
Using the formula for sum to infinity:

$$= \frac{a}{1-r}$$

Subbing in the values:

$$= \frac{8}{1-\frac{2}{3}}$$

Simplifying the terms

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$$= \frac{8}{\frac{1}{3}} = 24$$

33)

A geometric series, u_n has second term 375 and fifth term 81. Find the sum to infinity and hence the value of $\sum_{n=6}^{\infty} u_n$

We need to find a and r first

$$u_2 = 375 \text{ and } u_5 = 81$$
$$ar = 375, ar^4 = 81$$

Solve simultaneously

$$\frac{ar^4}{ar} = \frac{81}{375}$$
$$r^3 = \frac{27}{125}$$
$$r = \frac{3}{5} = 0.6$$
$$a = 625$$

There are 2 ways to proceed from here in order to find the sum

Way 1: Start the series from 6	**Way 2:** Force the series to start from 1
$\sum_{n=6}^{\infty} u_n$ $u_n = ar^{n-1}$ First term is $u_6 = 625(0.6)^5 = 48.6$ $S_\infty = \frac{48.6}{1-0.6} = 121.5$	$\sum_{n=6}^{\infty} u_n = \sum_{n=1}^{\infty} u_n - \sum_{n=1}^{5} u_n$ $= S_\infty - S_5$ $= \frac{625}{1-0.6} - \frac{625(1-0.6^5)}{1-0.6} = 1562.5 - 1441 = 121.5$

3.1 With Logs

34)

i.

$$\sum_{r=1}^{m} \ln 3^r$$

Start by writing the first 3 terms:

$$\ln 3^1 + \ln 3^2 + \ln 3^3 + \ldots$$

Use log rules to simplify

$$1 \ln 3 + 2 \ln 3 + 3 \ln 3 + \cdots$$

If it is an arithmetic series, the difference between the first and second is the same as between the second and third:

$$2 \ln 3 - 1 \ln 3 = \ln 3$$
$$3 \ln 3 - 2 \ln 3 = \ln 3$$

We have the difference the same here, therefore this is an arithmetic series with $a = \ln 3, d = \ln 3$

ii.

To start with simplify the summation:

first term $= a = \ln 3$
common difference $= d = \ln 3$

Using the summation formula

$$= \frac{n}{2}\left(2a + d(n-1)\right)$$

Subbing in the values

$$= \frac{20}{2}\left(2\ln 3 + \ln 3\,(20-1)\right)$$

$$= 10(2\ln 3 + 19\ln 3)$$

$$= 210\ln 3$$

iii.

Notice how this is the same premise as before, but instead of n being 20, n = 2m

Using the summation formula

$$= \frac{n}{2}\left(2a + d(n-1)\right)$$

Subbing in the values

$$= \frac{2m}{2}\left(2\ln 3 + \ln 3\,(2m-1)\right)$$

$$= m(2\ln 3 + (2m-1)\ln 3)$$

$$= m(2\ln 3 + 2mln\, 3 - \ln 3)$$

$$= m(\ln 3 + 2m\, ln\, 3)$$

$$= ln\, 3\, m + 2m^2 \ln 3$$

$$= \ln 3\,(m + 2m^2)$$

35)

$$\sum_{r=1}^{50} \ln(2^r)$$

i.
Start by writing the first few terms:

$$= \ln 2^1 + \ln 2^2 + \ln 2^3 + \cdots$$

Use log rules to simplify

$$= \ln 2 + 2\ln 2 + 3\ln 2 + \cdots$$

first term $= a$ = ln 2
common difference = d= ln 2
n = 50

This is arithmetic with a=ln2 and d=ln2 and we want s_{50}

Using the summation formula

$$= \frac{n}{2}\left(2a + d(n-1)\right)$$

Subbing in the values

$$= \frac{50}{2}\left(2\ln 2 + \ln 2\,(50-1)\right)$$

$$= 25(2\ln 2 + 49\ln 2)$$

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$$= 25(51\ln 2)$$
$$= 1275\ln 2$$

36)

i.

We are given the first 3 terms, so to work out the common ratio, it will be the second term divide by the first term:

$$= \frac{\ln x^8}{\ln x^{16}}$$

Using the laws of logs simplify

$$= \frac{8\ln x}{16\ln x}$$

Cancelling the common factors

$$= \frac{1}{2}$$

ii.

Start by writing out the first few terms:

$$= 2^{5-1}\ln x, 2^{5-2}\ln x, 2^{5-3}\ln x + \cdots$$

$$= 2^4\ln x + 2^3\ln x + 2^2\ln x + \cdots.$$

$$= 16\ln x, 8\ln x, 4\ln x$$

This is a geometric series with:
first term $=a = 16\ln x$
common ratio$= r = \frac{8\ln x}{16\ln x} = \frac{1}{2}$

We want to find the sum to infinity

Using the summation formula

$$= \frac{a}{1-r}$$

Subbing in the values

$$= \frac{16\ln x}{1-\frac{1}{2}}$$

$$= 32\ln x$$

Now setting this equal to 64:

$$32\ln x = 64$$

$$\ln x = 2$$

Raise both sides to the power of e

$$e^{\ln x} = e^2$$
$$x = e^2$$

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4 Diamond

4.1 With Logs

37)

i.

Start by simplifying the summation $\sum_{r=1}^{11} \ln p^n$

Using the laws of logs

$$\sum_{n=1}^{11} \mathrm{n} \ln p$$

Start by writing out the first few terms of the summation:

$$\sum_{n=1}^{11} \mathrm{n} \ln p$$

$$= \ln p + 2\ln p + 3\ln p + \cdots$$

This is an arithmetic series with:
first term = a = ln p
common difference= $d = 2\ln p - \ln p = \ln p$

Using the summation formula

$$= \frac{n}{2}(2a + d(n-1))$$

Subbing in the values

$$= \frac{11}{2}[2(\ln p) + \ln p\,(11-1)]$$

$$= \frac{11}{2}[2\ln p + 10\ln p]$$

$$= 5.5[12\ln p]$$

$$= 66\ln p$$

So

$$k = 66$$

ii.

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$$\sum_{n=1}^{11} \ln(8p^n)$$

Using the laws of logs $\ln(8p^n)$ can be split up as:

$$\ln(8p^n) = \ln 8 + \ln(p^n)$$

We can therefore write the summation formulas:

$$\sum_{n=1}^{11} \ln 8p^n = \sum_{n=1}^{11} \ln 8 + \sum_{n=1}^{11} \ln p^n$$

Working on the red first

This is just adding 11 lots of ln 8 since there is no variable inside the sum, doing that:

$$\sum_{r=1}^{11} \ln 8 = 11 ln\, 8$$

Working on the green next

$\sum_{n=1}^{11} \ln p^n = 66 \ln p$ from part i

$$\sum_{r=1}^{11} \ln 8p^n = 11 \ln 8 + 66 \ln p$$

Now we want to get the RHS in the form of the question:

$$11 \ln 8 = 11 \ln 2^3 = 33 \ln 2$$

$$66 \ln p = (33 \times 2) \ln p = (2 \times 33) \ln p = 33 \ln p^2$$

Adding the terms:

$$= 33 \ln 2 + 33 \ln p^2$$

$$33(\ln 2 + \ln p^2)$$

$$= 33 \ln 2p^2$$

iii.

Using the answer to part ii:

$$33 \ln 2p^2 < 0$$

Divide both sides by 33

$$\ln 2p^2 < 0$$

Raise both sides to the power of e:

$$2p^2 < 1$$

Rearrange for p^2

$$p^2 < \frac{1}{2}$$

Use a number line or graph to solve the inequality

$$-\frac{\sqrt{2}}{2} < p < \frac{\sqrt{2}}{2}$$

but p can't be negative since told p is a positive constant

$$0 < p < \frac{\sqrt{2}}{2}$$

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38)

$$\sum_{n=1}^{15} {a_n}^2 = lnx^n$$

Write out a few terms

$$(lnx^1)^2 + (lnx^2)^2 + (lnx^3)^2 + \cdots + (lnx^{15})^2$$

$$= (\ln x)^2 + (2lnx)^2 + (3lnx)^2 + \cdots + (15lnx)^2$$

We can't use log rules. We can only bring the power up and down if it is the power of the argument of ln. We can't bring the power down when the whole ln is to that power

This sum is neither arithmetic or geometric (it is a quadratic sequence) and we have no real way of solving this, rather then working out all of the terms:

$$= (\ln x)^2 + 2^2(\ln x)^2 + 3^2(\ln x)^2 + \cdots + 15^2(lnx)^2$$

$$= (lnx)^2[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots + 15^2]$$

$$= 1240(lnx)^2$$

39)

Start by working out the first few terms:

$$\sum_{n=1}^{48} \log_5\left(\frac{n+2}{n+1}\right)$$

$$= \log_5\frac{3}{2} + \log_5\frac{4}{3} + \log_5\frac{5}{4} + \log_5\frac{6}{5} + \cdots + \log_5\frac{50}{49}$$

This is neither a geometric not arithmetic series. We will have to use log rules to try and simplify instead

We can proceed in 2 different ways.

$$\log_5\frac{3}{2} + \log_5\frac{4}{3} + \log_5\frac{5}{4} + \log_5\frac{6}{5} + \cdots + \log_5\frac{50}{49}$$

Use log rules: $\log a + \log b = \log ab$

$$\log\left(\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times\frac{6}{5}\times\ldots\times\frac{48}{47}\times\frac{49}{48}\times\frac{50}{49}\right)$$

$$\log\left(\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times\frac{6}{5}\times\ldots\times\frac{48}{47}\times\frac{49}{48}\times\frac{50}{49}\right)$$

all will cancel out except the first and the last (see the colour pairs)

$$= \log_5\left(\frac{50}{2}\right)$$

$$= \log_5 25$$

Writing this as an integer (simplifying)

$$= 2$$

$$\log_5\frac{3}{2} + \log_5\frac{4}{3} + \log_5\frac{5}{4} + \log_5\frac{6}{5} + \cdots + \log_5\frac{50}{49}$$

$$\log_5\frac{3}{2} + \log_5\frac{4}{3} + \log_5\frac{5}{4} + \log_5\frac{6}{5} + \cdots + \log_5\frac{50}{49}$$

Use log rules: $\log\frac{a}{b} = \log - \log b$

$$(\log_5 3 - \log_5 2) + (\log_5 4 - \log_5 3) + (\log_5 5 - \log_5 4) + (\log_5 6 - \log_5 5) + \cdots + (\log_5 50 - \log_5 49) +$$

All terms cancel except the first and the last

$$= -\log_5 2 + \log_5 50$$

$$= \log_5 50_5 - \log_5 2$$

$$= \log_5 25$$

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	Writing this as an integer (simplifying) $= 2$

40)

Start by writing out all of the terms (here we use n to represent the infinity term)

$$S = \left[\frac{1}{2} - \frac{1}{3}\right] + \left[\frac{1}{3} - \frac{1}{4}\right] + \left[\frac{1}{4} - \frac{1}{5}\right] + \cdots + \left[\frac{1}{n-1} - \frac{1}{n}\right] + \left[\frac{1}{n} - \frac{1}{n+1}\right] + \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

Notice how apart from the two red terms, ever other one has an opposite pair, we can use these to cause cancelations

$$= \frac{1}{2} + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \left(-\frac{1}{5} + \frac{1}{5}\right) + \cdots\left(-\frac{1}{n-1} + \frac{1}{n-1}\right) + \left(-\frac{1}{n} + \frac{1}{n}\right) + \left(-\frac{1}{n+1} + \frac{1}{n+1}\right) - \frac{1}{n+2}$$

Simplifying:

$$= \frac{1}{2} - \frac{1}{n+2}$$

4.2 Two Series

41)

Hint: There are 2 series here. Use fact that $\sum x + y = \sum x + \sum y$

Writing this in sigma notation

$$\sum_{r=1}^{16} 3 + 5\mathrm{r} + 2^{\mathrm{r}}$$

Splitting this up into 2 series

$$\sum_{r=1}^{16} 3 + 5\mathrm{r} \ + \sum_{r=1}^{16} 2^r$$

Looking at the blue equations:

$$= \big(3 + 5(1)\big) + \big(3 + 5(2)\big) + \big(3 + 5(3)\big) + \cdots$$

$$= 8 + 13 + 18 + \cdots$$

This is an arithmetic series with:
first term = $a = 8$
common difference= $d = 5$
we have 16 terms

Using the sum equation:

$$= \frac{n}{2}(2a + d(n-1))$$

$$= \frac{16}{2}(2(8) + 5(16-1))$$
$$= 728$$

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Now looking at the green sum

Writing out the first few terms:

$$= 2^1 + 2^2 + 2^3 \ldots$$
$$= 2 + 4 + 8$$

This is a geometric series with:
first term = $a = 8$
common ratio= $r = 2$
we have 16 terms

Using the equation:

$$= \frac{a(1-r^n)}{1-r}$$

$$= \frac{2(1-2^{16})}{1-2}$$

$$= 131070$$

Adding the two series:

$$= 728 + 131070 = 131798$$

42)

Hint: There are 2 series here. Use fact that $\sum x + y = \sum x + \sum y$)

Writing this in sigma notation

$$\sum_{n=0}^{\infty} \frac{2^n + 4^n}{6^n}$$

Splitting this up into 2 series

$$\sum_{r=0}^{\infty} \frac{2^n}{6^n} + \sum_{r=0}^{\infty} \frac{4^n}{6^n}$$

Looking at the blue series:

Write the first few terms

$$= \frac{2^0}{6^0} + \frac{2^1}{6^1} + \frac{2^2}{6^2} + \cdots$$

$$= 1 + \frac{1}{3} + \frac{1}{9} + \cdots$$

This is a geometric series with:
first term = $a = 1$
common ratio= $r = \frac{1}{3}$
We have an infinite number or terms so we can use the sum to infinity

Using the sum to infinity equation:

$$= \frac{1}{1-\frac{1}{3}}$$

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$$=\frac{1}{\frac{2}{3}}$$

$$=\frac{3}{2}$$

Now looking at the green series

Writing out the first few terms:

$$=\frac{4^0}{6^0}+\frac{4^1}{6^1}+\frac{4^2}{6^2}+\cdots$$
$$=1+\frac{2}{3}+\frac{4}{9}+\cdots$$

This is a geometric series with:
first term = $a=1$
common ratio= $r=\frac{2}{3}$
We have an infinite number or terms so we can use the sum to infinity

Using the sum to infinity equation:

$$=\frac{a}{1-r}$$

$$=\frac{1}{1-\frac{2}{3}}$$
$$=\frac{1}{\frac{1}{3}}$$

$$=3$$

Adding the two series:

$$=\frac{3}{2}+3=\frac{9}{2}$$

43)

$$\sum_{r=1}^{10} a+(r-1)d=\sum_{r=11}^{14} a+(r-1)d$$

Notice how both arithmetic series with just the nth term formulas so we can use

Working on the LHS:
this is an arithmetic series with
first term = a
common difference= d
number of terms= 10

Using this information, we can use the summation formula:

$$=\frac{n}{2}(2a+d(n-1))$$

$$=\frac{10}{2}(2a+d(10-1))$$

$$=5(2a+9d)$$

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$$= 10a + 45d$$

Working on the RHS now:

This is just the sum of 4 terms. As there is just 4 terms, we can just directly find them

$$= a + 10d + a + 11d + a + 12d + a + 13d$$

Collecting like terms:

$$= 4a + 46d$$

Equating the two terms:

$$10a + 45d = 4a + 46d$$

$$d = 6a$$

44)

i.

$$s_{15} = 1320$$
$$u_5 = 76$$

We can use these equations with their relevant formulae

$$\frac{15}{2}\big(2a + d(15-1)\big) = 1320$$
$$\frac{15}{2}(2a + 14d) = 1320$$
$$\frac{15}{2}(2a + 14d) = 1320$$

$$2a + 14d = 176$$

$$7a + d(5-1) = 76$$
$$a + 4d = 76$$

Now solving the equations simultaneously:

$$176 = 2a + 14d$$
$$76 = a + 4d$$

$$a = 60$$
$$d = 4$$

ii.

We have been given

$$13\left(\sum_{n=1}^{15} u_n - \sum_{n=1}^{k} u_n\right) = 9\sum_{n=1}^{k} u_k$$

We want to get rid of all of the summations:

From part i, we know that the sum of the first 15 terms is 1320

$$13\left(1320 - \sum_{n=1}^{k} u_n\right) = 9\sum_{n=1}^{k} u_k$$

Now expanding the backets

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$$17160 - 13\sum_{n=1}^{k} u_n = 9\sum_{n=1}^{k} u_k$$

Group the summations on one side (since they're the same)

$$22\sum_{n=1}^{k} u_k = 17160$$

Simplify

$$\sum_{n=1}^{k} u_k = 780$$

$u_k = a + (k-1)d = 60 + (k-1)4 = 56 + 4n$

Replacing u_k with $56 + 4n$

$$\sum_{n=1}^{k} (56 + 4n) = 780$$

Doing the sum, using the summation formula:

$$= \frac{n}{2}\big(2a + d(n-1)\big)$$

Working on the LHS
this is an arithmetic series with
first term = $a = 60$
common difference= $d = 4$
number of terms= k

$$\frac{k}{2}\big(120 + 4(k-1)\big) = 780$$

$$\frac{k}{2}(4k + 116) = 780$$

Simplifying

$$k(4k + 116) = 1560$$

$$4k^2 + 116k - 1560 = 0$$

$$k = 10, -39$$

k must be positive, so k = 10

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5 Challenges

5.1 Arithmetic

45)

i.

The 25th term is 100, therefore:

$$100 = a + d(25 - 1)$$
$$100 = a + 24d$$

Now using the second line:
The 5th term:

$$= a + 4d$$

The 35th term:

$$= a + 34d$$

So using the fact that 5th term is 8* the 35th term:

$$8(a + 34d) = a + 4d$$
$$8a + 272d = a + 4d$$
$$7a + 268d = 0$$

Now working with the simultaneous equations

$$7a + 268d = 0$$
$$100 = a + 24d$$

$$a = \ 268\,, d = \ -7$$

ii.
Using the formula for the nth term:

$$= a + d(n - 1)$$

$$= 268 - 7(n - 1)$$
$$= 275 - 7n$$

Now working out now many terms are more then 0

$$275 - 7n > 0$$
$$275 > 7n$$
$$\frac{275}{7} > n$$
$$39.29 > n$$
$$So\ 39\ terms$$

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iii.

This question states the maximum, which means we need to differentiate

$$\sum_{n=1}^{k} 56 + 4n$$

Doing the sum, using the summation formula:

$$s_n = \frac{n}{2}\big(2a + d(n-1)\big)$$

a = 268
n = k
d = −7

$$s_n = \frac{k}{2}\big(536 - 7(k-1)\big)$$

$$s_n = \frac{k}{2}(-7k + 543)$$

Simplifying

$$s_n = -\frac{7}{2}k^2 + \frac{543}{2}k$$

Differentiating

$$derivative = -7k + \frac{543}{2}$$

Setting equal to zero

$$-7k + \frac{543}{2} = 0$$

$$k = \frac{543}{14}$$

Subbing that value in

$$s_n = -\frac{7}{2}\left(\frac{543}{14}\right)^2 + \frac{543}{2}\left(\frac{543}{14}\right)$$

$$s_n = 5265$$

46)

Writing this as the sum of 2 sums:

$$\left(\sum_{n=1}^{20} 2r + \sum_{n=1}^{20} x\right) = 280$$

Looking at the green

$$\sum_{n=1}^{20} x$$

We know that this is just the sum of 20 x, therefore:

$$\sum_{n=1}^{20} x = 20x$$

Now working on the red

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$$\sum_{n=1}^{20} 2r$$

Start by writing out the terms of the equation:

$$= 2(1) + 2(2) + 2(3) \ldots$$

$$= 2 + 4 + 6 \ldots$$

Doing the sum, using the summation formula:

$$s_n = \frac{n}{2}\big(2a + d(n-1)\big)$$

a = 2
n = 20
d = 2

$$s_n = \frac{20}{2}\big(2(2) + 2(20-1)\big)$$
$$s_n = \frac{20}{2}\big(2(2) + 2(20-1)\big)$$

$$s_n = 420$$

Therefore:

$$420 + 20x = 280$$
$$x = -7$$

47)

i.

First of all writing this is a form we can solve for

$$\frac{5}{2}(5k + 28) = 370$$

Rearrange for n

$$(5k + 28) = 148$$

$$5k = 120$$

$$k = 24$$

ii.
Subbing in k for the summation and expanding the nth term

$$\sum_{n=1}^{24} \left(\frac{25}{2}n + 70\right)$$

Writing the first few terms:

$$= \frac{165}{2} + 95 + \frac{215}{2}$$

Doing the sum, using the summation formula:

$$s_n = \frac{n}{2}\big(2a + d(n-1)\big)$$

a = $\frac{165}{2}$
n = 24
d = $\frac{25}{2}$

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$$s_n = \frac{24}{2}\left(2\left(\frac{165}{2}\right) + \frac{25}{2}(24-1)\right)$$

$$= 5430$$

48)

Writing this as the sum of 2 sums (for the green, we factor out p, as it is a constant):

$$\left(\sum_{n=1}^{20} 25 + p\sum_{n=1}^{20} n\right) = 80$$

Looking at the green

$$p\sum_{n=1}^{20} n$$

Looking at the sum, write the first few terms

$$= 1 + 2 + 3 + 4$$

Doing the sum, using the summation formula:

$$= \frac{n}{2}(2a + d(n-1))$$

Working on the LHS first

The constants =

a = $\frac{1}{2}$

n = 20

d = 1

$$\frac{20}{2}(2(1) + 1(20-1))$$

$$= 210$$

We initially factored out a p, putting that back in:

$$= 210p$$

Now working on the red

$$\sum_{n=1}^{20} 25$$

This is the same as doing 25 lots of 20 as we add 25, 20 times:

$$= 25 * 20 = 500$$

Writing this s an equation

$$500 + 210p = 80$$

$$p = -2$$

49) A sequence is defined as $u_{r+1} = u_r - 3$, $u_{1=117}$

Solve the equation $\sum_{n=1}^{n} u_r = 0$

the first term is 117

Looking at the formula, the second term is 117 – 3 = 114 (since $u_2 = u_1 - 3$)

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The third term is 114 – 3 = 111

Doing the sum, using the summation formula:

$$=\frac{n}{2}\big(2a+d(n-1)\big)$$

The constants =
a = 117
n = we want to work it out
d = −3

$$0=\frac{n}{2}\big(2(117)-3(n-1)\big)$$
$$0=\frac{n}{2}(-3n+237)$$

This kind of looks like a quadratic, so we want one of the brackets to be 0, so using that information

$$-3n+237=0$$

$$n=79$$

50) Find in simplified for the terms of n, the value of

$$\sum_{n=1}^{2n}(3r-2)(-1)^r$$

Looking at the equation, $(-1)^r$ is a “swapper term”, it swaps the sign from positive to negative:

Start by writing out the first few terms of the sequence (we may need more than normal to establish a patten)

$$=(-1)^1(3(1)-2)+(-1)^2(3(2)-2)+(-1)^3(3(3)-2)+\ (-1)^4(3(4)-2)+\ (-1)^5(3(5)-2)+\ (-1)^6(3(6)-2)$$

$$=-1+4+\ -7+10-13+16$$

If you look at the numbers closely we notice we have 2 sequences:

$$=(-1-7-13)+(4+10+16)$$

Both of those sequences are arithmetic, as we can see from the constant difference

Furthermore, notice how the question has the sum of the first 2n terms, this means the sum of the first n terms of these 2 sequences:

Working on the blue sequence

Doing the sum, using the summation formula:

$$=\frac{n}{2}\big(2a+d(n-1)\big)$$

a = −1
n = n
d = −6

$$0=\frac{n}{2}\big(2(-1)-6(n-1)\big)$$
$$=\frac{n}{2}(-6n+4)$$
$$=-3n^2+2n$$

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Working on the pink sequence

Doing the sum, using the summation formula:

$$=\frac{n}{2}\big(2a+d(n-1)\big)$$

a = 4
n = we want to work it out
d = 6

$$=\frac{n}{2}\big(2(4)+6(n-1)\big)$$
$$=\frac{n}{2}(6n+2)$$
$$=3n^2+n$$

Adding the terms

$$=-3n^2+2n+3n^2+n$$

$$=3n$$

51)

We have been given the nth term using that:

We can split this into 2 summations (to start from 1 which is easier)

$$\sum_{r=N}^{3N} u_r=\sum_{r=1}^{3N}(120-3r)-\sum_{r=1}^{N-1}(120-3r)$$

Let's deal with the blue sequence first
Start by writing out the first few terms of the sequence

$$=120-3(1)+120-3(2)+120-3(3)+\cdots$$
$$=117+114+111+\cdots$$

We can see that
a = 117
n = 3N
d = −3

Using the summation formula:

$$=\frac{n}{2}\big(2a+d(n-1)\big)$$

$$=\frac{3N}{2}\big(2(117)-3(3N-1)\big)$$
$$=\frac{3N}{2}(-9N+237)$$

Let's deal with the red sequence next

Start by writing out the first few terms of the sequence

$$=120-3(1)+120-3(2)+120-3(3)+\cdots$$
$$=117+114+111+\cdots$$

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Doing the sum, using the summation formula:

$$=\frac{n}{2}\left(2a+d(n-1)\right)$$

We can see that
a = 117
n = N−1
d = −3

$$=\frac{N-1}{2}\left(2(117)-3(N-2)\right)$$
$$=\frac{N-1}{2}(-3N+240)$$

Equating the terms

$$\frac{3N}{2}(-9N+237)-\frac{N-1}{2}(-3N+240)=444$$

Now expanding and solving

$$3N(-9N+237)-(N-1)(-3N+240)=888$$

$$-27N^2+711N-(-3N^2+240N+3N-240)=888$$

$$-27N^2+711N-(-3N^2+243N-240)=888$$

$$-27N^2+711N+3N^2-243N+240=888$$

$$-24N^2+468N-648=0$$

$$n=18,\frac{3}{2}$$

We cannot have a fraction

Therefore

$$n=18$$

52)

We have two sums, splitting up into 2 sums

$$\sum_{r=1}^{2k}u_n-\sum_{r=1}^{k}u_n$$

Doing the sum, using the summation formula:

$$=\frac{n}{2}\left(2a+d(n-1)\right)$$

The constants (we have been given the constants in the question)=
a = -10
n = 2k
d = 4

$$=\frac{2k}{2}\left(2(-10)+4(2k-1)\right)$$
$$=k(8k-24)$$

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Doing the sum, using the summation formula:

$$=\frac{n}{2}\left(2a+d(n-1)\right)$$

The constants =
a = -10
n = k
d = 4

$$=\frac{k}{2}\left(2(-10)+4(k-1)\right)$$
$$=\frac{k}{2}(4k-24)$$
$$=k(2k-12)$$

Writing the equation:

$$k(8k-24)-\frac{k}{2}(4k-24)=1728$$

Expand and solve:

$$k(8k-24)-\ k(2k-12)=1728$$

$$8k^2-24k-2k^2+12k=1728$$

$$6k^2-12k-1728=0$$

$$k=18,-16$$

K cannot be negative

$$k=18$$

53)

i.

The sum of the first 25 terms is 1050, subbing this into the summation formula

$$=\frac{n}{2}\left(2a+d(n-1)\right)$$

Working on the LHS first

The constants =
a = a
n = 25
d = d

$$1050\ =\frac{25}{2}\left(2a+d(25-1)\right)$$
$$1050\ =\frac{25}{2}(2a+24d)$$

$$84\ =2a+24d$$

Also the 25th term is 72, using this equation:

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$$72 = a + d(25-1)$$
$$72 = a + 24d$$

Now solving the equations:

$$84 = 2a + 24d$$
$$72 = a + 24d$$

$$a = 12$$
$$d = 2.5$$

ii.

We have been given

$$117\left(\sum_{n=1}^{25} u_n - \sum_{n=1}^{k} u_n\right) = 233\sum_{n=1}^{k} u_k$$

So we want to get rid of all of the summations:

To do so, working on the colours:

From part I, we know that the sum of the first 15 terms is 1320

$$117\left(1050 - \sum_{n=1}^{k} u_n\right) = 233\sum_{n=1}^{k} u_k$$

Now expanding the backets

$$122850 = 350\sum_{n=1}^{k} u_k$$

Simplify

$$351 = \sum_{n=1}^{k} u_k$$

Replacing $u_k = 2.5x + \frac{19}{2}$

$$351 = \sum_{n=1}^{k} 2.5n + \frac{19}{2}$$

Doing the sum, using the summation formula:

$$= \frac{n}{2}(2a + d(n-1))$$

Working on the LHS first

The constants =
a = 12
n = k
d = 2.5

$$351 = \frac{k}{2}(2(12) + 2.5(k-1))$$
$$351 = \frac{k}{2}(2.5k + 21.5)$$

Simplifying

$$351 = \frac{5}{4}k^2 + \frac{43}{4}k$$

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$$0 = \frac{5}{4}k^2 + \frac{43}{4}k - 351$$

$$k = 13, -\frac{108}{5}$$

So k must be positive, so k = 13

54)

We have two sums, splitting up into 2 sums

$$\sum_{r=K+1}^{N} 4r - 7 - \sum_{r=1}^{K} 4r - 7$$

We can split the blue up into 2 terms as well

$$\sum_{r=K+1}^{N} 4r - 7 = \sum_{r=1}^{N} 4r - 7 - \sum_{r=1}^{K} 4r - 7$$

So now putting back to the original equation:

$$\sum_{r=1}^{N} 4r - 7 - \sum_{r=1}^{K} 4r - 7 - \sum_{r=1}^{K} 4r - 7$$

$$\sum_{r=1}^{N} 4r - 7 - 2\sum_{r=1}^{K} 4r - 7$$

Doing the sum, using the summation formula:

$$= \frac{n}{2}(2a + d(n-1))$$

The constants (we have been given the constants in the question) =
A =−-3
n = N
D = 4

$$= \frac{N}{2}(2(-3) + 4(N-1))$$
$$= \frac{N}{2}(-6 + 4N - 4)$$
$$= \frac{N}{2}(4N - 10)$$
$$= 2N^2 - 5N$$

Now doing the sum of the red, notice how it is the name formula, but with a K rather then N:

$$= 2K^2 - 5K$$

Now simplifying:

$$400 = 2N^2 - 5N - 2(2K^2 - 5K)$$
$$400 = 2N^2 - 5N - 4K^2 + 10K$$

Now using the second equation

$$u_N = a + d(N-1)$$

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Subbing the values:

$$u_N = -3 + 4(N-1)$$
$$u_N = 4N - 7$$

$$u_K = -3 + 4(K-1)$$

$$u_K = 4k - 7$$

Now putting that into equation form

$$4N - 7 - 4K + 7 = 40$$
$$N - K = 10$$

Now we have 2 equations

$$400 = 2N^2 - 5N - 4K^2 + 10K$$
$$N - K = 10$$

Now rearrange equation 2:

$$N = 10 + k$$

But into the longer equation

$$400 = 2(10+k)^2 - 5(10+k) - 4K^2 + 10K$$

Expand the brackets

$$400 = 2k^2 + 40k + 200 - 50 - 5k - 4K^2 + 10K$$

$$0 = -2k^2 + 45k - 250$$

$$k = 10, \frac{25}{2}$$

K can not be a decimal, therefore

$$k = 10$$

Now sub into the linear equation

$$N - 10 = 10$$

$$N = 20$$

5.2 Geometric

55)

The sum of the geometric series is 2187 means that $s_\infty = 2187$

$$\frac{a}{1-r} = 2187$$

$$a = 2187(1-r)$$

$$a + 2187r = 2187$$

We have been given 2 consecutive terms, using these we can work out the common ratio:

$$ratio = \frac{u_k}{u_{k-1}} = \frac{64}{96} = \frac{2}{3}$$

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Subbing that into the equation

$$2187 = a + 2187\left(\frac{2}{3}\right)$$

$$a = 729$$

Now the question says the sum from the k+1 to infinity

$$\text{k+1 term} = 64\left(\frac{2}{3}\right) = \frac{128}{3}$$

So, using the sum to infinity formula with the k+1 term being the first term:

$$= \frac{\left(\frac{128}{3}\right)}{1 - \frac{2}{3}}$$

$$= 128$$

56)

i.

We have been given a formula for the first n terms, and we want to work out the value of the sum of the first 8 terms

$$= 128 - 2^{7-8}$$

$$= 127.5$$

ii.
To work out the value the 8th term, we can do the sum of the first 7 terms, and then the sum of the first 8 terms, and subtracting them will give you the 8th term

$$= 127.5 - (128 - 2^{7-7})$$

$$= 0.5$$

iii.

Following the theme of the other questions, try and work out the 7th term by doing the sum of the 7 – sum of 6:

$$(128 - 2^{7-7}) - (128 - 2^{7-6})$$

$$= 1$$

Now working out the ratio

$$= \frac{0.5}{1} = \frac{1}{2}$$

57)

Working out the first few terms:

$$= 2^{2(1)-1} + 2^{2(2)-1} + 2^{2(3)-1}$$

$$= 2 + 8 + 32$$

Working out the common ratio

$$= \frac{8}{2} = 4$$

So using the formula for the sum of n terms:

$$= \frac{a(1 - r^n)}{1 - r}$$

For the constants:
A = 2
R = 4

$$43690 = \frac{2(1 - 4^n)}{1 - 4}$$

Rearrange

$$43690 = \frac{2(1 - 4^n)}{-3}$$
$$43690 = \frac{2(1 - 4^n)}{-3}$$

$$-65535 = 1 - 4^n$$

$$65536 = 4^n$$

$$n = 8$$

58)

We are given that the first term is 1458

We are also given that the 6th term is 6 ($U_6 = 6$), therefore we use the formula:

$$u_n = ar^{n-1}$$

Subbing into the equation

$$6 = 1458r^{6-1}$$

Rearrange for r

$$6 = 1458r^5$$
$$r = \frac{1}{3}$$

Now we want to find $\sum_{n=7}^{\infty} u_n$

So, we need to use the equation

$$Sum\ to\ infinity = \frac{a}{1 - r}$$

We know that

$$a = 1458\left(\frac{1}{3}\right)^{7-1} = 2$$

$$r = \frac{1}{3}$$

$$Sum\ to\ infinity = \frac{2}{1 - \frac{1}{3}}$$

$$Sum\ to\ infinity = 3$$

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59)

S_k represents the sum to infinity. So we need to use the equation

$$\text{Sum to infinity} = \frac{a}{1-r}$$

Starting with this, and subbing in the values
A = $\frac{k-1}{k!}$
ratio = $\frac{1}{k}$

Subbing this into the formula

$$= \frac{\frac{k-1}{k!}}{1-\frac{1}{k}}$$

$$= \frac{\frac{k-1}{k!}}{\frac{k-1}{k}}$$

Factoring out the common factors

$$= \frac{k}{k!}$$

$$= \frac{1}{(k-1)!}$$

So now putting this into the summation:

$$\sum_{k=3}^{100} \left(\left((k-1)(k-2)-1\right)\frac{1}{(k-1)!}\right)$$

This looks hard so first try and find a patten:

Expanding the brackets in the summation

$$\sum_{k=3}^{100} \left(\frac{1}{(k-3)!} - \frac{1}{(k-1)!}\right)$$

So now writing out the terms:

$$\frac{1}{(3-3)!} - \frac{1}{(3-1)!}$$
$$\frac{1}{(4-3)!} - \frac{1}{(4-1)!}$$
$$\frac{1}{(5-3)!} - \frac{1}{(5-1)!}$$

Simplifying:

$$\frac{1}{(0)!} - \frac{1}{(2)!}$$
$$\frac{1}{(1)!} - \frac{1}{(3)!}$$

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$$\frac{1}{(2)!} - \frac{1}{(4)!}$$

Looking at the red, notice the cancellations

By the process of eliminations, we see that the green terms are the only one which wont be cancelled out from here

By symmetry this implies that the last two items will also not cancel, therefore:

$$\sum_{k=3}^{100} \left((k-1)(k-2)-1)\frac{1}{(k-1)!}\right) = \frac{1}{0} + \frac{1}{1} - \frac{1}{98!} - \frac{1}{99!}$$

Now writing in the form of the question

$$= \frac{10^4}{100!} + \frac{1}{1} + \frac{1}{1} - \frac{1}{98!} - \frac{1}{99!}$$

Simplifying because our calculators cant do this:

$$= \frac{10^4}{100!} + \frac{2(100!)}{100!} - \frac{(100)(99)}{100!} - \frac{100}{100!}$$
$$= \frac{10^4 + 2(100!) - (100)(99) - 100}{100!}$$

Factoring out 100

$$= \frac{100(100 + 2(99!) - (99) - 1)}{100!}$$
$$= \frac{100(2(99!))}{100!}$$

Putting the 100 back in

$$= \frac{(2(100!))}{100!}$$

Factor out the 100!

$$= 2$$

60)

Let's start by writing out the first few terms:

$$(1+0) \times 11^0 \times 10^9 = 1 \times 10^9$$
$$(1+1) \times 11^1 \times 10^{9-1} = 2 \times 11 \times 10^8$$
$$(2+1) \times 11^2 \times 10^{9-2} = 3 \times 11^2 \times 10^7$$
$$(3+1) \times 11^3 \times 10^{9-3} = 4 \times 11^3 \times 10^6$$

As we can see this it like a patten:

$$S = 1(11)^0(10)^9 + 2(11)^1(10)^8 \ldots + 9(11)^8(10)^1 + 10(11)^9(10)^0$$

We need to make this look like a geometric sequence, so take out a factor of $(10)^9$

$$S = 10^9\left(1 + 2\left(\frac{11}{10}\right)^1 \ldots + 9\left(\frac{11}{10}\right)^8 + 10\left(\frac{11}{10}\right)^9\right)$$

This sequence has both arithmetic and geometric features, so we multiply by $-\frac{11}{10}$

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$$-\frac{11}{10}S = 10^9(-\frac{11}{10} - 2\left(\frac{11}{10}\right)^2 \ldots - 9\left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10})$$

add the following together

$S = 10^9(1 + 2\left(\frac{11}{10}\right)^1 \ldots + 9\left(\frac{11}{10}\right)^8 + 10\left(\frac{11}{10}\right)^9)$ and $-\frac{11}{10}S = 10^9(-\frac{11}{10} - 2\left(\frac{11}{10}\right)^2 \ldots - 9\left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10})$

This gives

$$(1 - \frac{11}{10})S = 10^9\left(1 + \frac{11}{10} + \left(\frac{11}{10}\right)^2 \ldots + \left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10}\right)$$

We need to use the equation

$$s_n = \frac{a(1 - r^n)}{1 - r}$$

a = 1
$r = \frac{11}{10}$
n = 10

$$-\frac{1}{10}S = 10^9 \times \frac{\left(\frac{11}{10}\right)^{10} - 1}{\frac{11}{10} - 1} + 10^9(-10 \times \left(\frac{11}{10}\right)^{10})$$

$$-\frac{1}{10}S = 10^{10}\left(\left(\frac{11}{10}\right)^{10} - 1\right) - 10^{10}(\left(\frac{11}{10}\right)^{10})$$

Expanding and simplifying

$$-\frac{1}{10}S = -10^{10}$$
$$\frac{1}{10}S = 10^{10}$$

$$S = 10^{11}$$

61)

Lets start by writing out the first few terms to establish a pattern:

$$= (1 + n)^{-2} + (1 + n)^{-3} + +(1 + n)^{-4}$$

This is a sum to infinity, so subbing into the formula:

$$Sum\ to\ infinity = \frac{a}{1 - r}$$

The constants =
a = $(1 + n)^{-2}$
d = $(1 + n)^{-1}$

$$\frac{27}{4}n = \frac{(\frac{1}{(1 + n)^2})}{1 - \frac{1}{1 + n}}$$

Simplifying:

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$$\frac{27}{4}n = \frac{\left(\frac{1}{(1+n)^2}\right)}{\left(\frac{1+n-1}{1+n}\right)}$$

$$\frac{27}{4}n = \frac{\left(\frac{1}{(1+n)^2}\right)}{\left(\frac{n}{1+n}\right)}$$

Simplifying:

$$\frac{27}{4}n = \frac{1}{n(n+1)}$$

Rearrange for a cubic

$$\frac{27}{4}n^2(n+1) = 1$$

$$\frac{27}{4}n^3 + \frac{27}{4}n^2 - 1 = 0$$

$$n = \frac{1}{3}, -\frac{2}{3}$$

But n can not be $-\frac{2}{3}$

$$n = \frac{1}{3}$$

62)

This is a sum embedded into a sum, so work on each one individually

Starting with the inner sum:

Write out the first few terms:

$$= \frac{1}{2^{0+n}} + \frac{1}{2^{1+n}} + \frac{1}{2^{2+n}}$$

This is a sum of a geometric series, so we need the formula

$$sum = \frac{a(1-r^n)}{1-r}$$

The constants are:

a = $\frac{1}{2^n}$

r = $\frac{1}{2}$

n = n

$$sum = \frac{\frac{1}{2^n}\left(1-\left(\frac{1}{2}\right)^{n+1}\right)}{1-\frac{1}{2}}$$

Now we have a second sum:

$$2\sum_{n=0}^{\infty} \frac{1}{2^n} - \frac{1}{2^{2n+1}}$$

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Splitting this up into 2 sums

$$2\sum_{n=0}^{\infty}\frac{1}{2^n}-2\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}$$

Writing out some of the terms

$$=2\left(1+\frac{1}{2}+\frac{1}{4}+\ldots\right)-2\left(\frac{1}{2}+\frac{1}{8}+\frac{1}{32}\ldots\right)$$

Looking at the purple first

Using:

$$sum\ to\ infintiy=\frac{a}{1-r}$$

$$sum\ to\ infintiy=\frac{1}{1-\frac{1}{2}}$$

$$sum\ to\ infintiy=2$$

$$sum\ to\ infintiy=2\times 2=4$$

Now looking at the blue

$$sum\ to\ infintiy=\frac{a}{1-r}$$

$$sum\ to\ infintiy=\frac{\frac{1}{2}}{1-\frac{1}{4}}$$

$$sum\ to\ infintiy=\frac{2}{3}$$

$$sum\ to\ infintiy=2\times\frac{2}{3}=\frac{4}{3}$$

So subtracting the terms

$$4-\frac{4}{3}=\frac{8}{3}$$

63)

This is a sum embedded into a sum, so work on each one individually

Starting with the inner sum:

Write out the first few terms:

$$=\frac{1}{3^{0+n}}+\frac{1}{3^{1+n}}+\frac{1}{3^{2+n}}$$

This is a sum of a geometric series, so we need the formula

$$sum=\frac{a}{1-r}$$

The constants are:

a = $\frac{1}{3^n}$

r = $\frac{1}{3}$

n = n

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$$sum = \frac{\frac{1}{3^n}}{1-\frac{1}{3}}$$

Now we have a second sum:

$$\sum_{n=0}^{\infty} \frac{3}{2} * \frac{1}{3^n}$$
$$\frac{3}{2}\sum_{n=0}^{\infty} \frac{1}{3^n}$$

Using:

$$sum\ to\ infintiy = \frac{a}{1-r}$$

a = 1
r = $\frac{1}{3}$

$$sum\ to\ infintiy = \frac{1}{1-\frac{1}{3}}$$
$$sum\ to\ infintiy = \frac{3}{2}$$

Now

$$\frac{3}{2} \times \frac{3}{2} = \frac{9}{4}$$

64) It is given that the following series converges to the limit L.

$$\sum_{r=1}^{\infty} \left[\frac{2x-1}{x+2}\right]^r$$

Determine with full justification the range of possible values of L.

Start by writing out some of the terms:

$$\left(\frac{2x-1}{x+2}\right)^1 + \left(\frac{2x-1}{x+2}\right)^2 + \left(\frac{2x-1}{x+2}\right)^3 + \cdots$$

Using:

$$sum\ to\ infintiy = \frac{a}{1-r}$$

$a = \frac{2x-1}{x+2}$

r = $\frac{2x-1}{x+2}$

$$sum\ to\ infintiy = \frac{\frac{2x-1}{x+2}}{1-\frac{2x-1}{x+2}}$$

Simplifying

$$sum\ to\ infintiy = \frac{\frac{2x-1}{x+2}}{\frac{(x+2)-(2x-1)}{x+2}}$$

$$sum\ to\ infintiy = \frac{\frac{2x-1}{x+2}}{\frac{3-x}{x+2}}$$

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Simplifying

$$sum\ to\ infintiy = \frac{2x-1}{3-x}$$

So now we want to find the rage of values for this:

So as we know |r|<1
Therefore:

$$\frac{2x-1}{x+2} < 1$$
$$2x-1 < 1(x+2)$$
$$x < 3$$

$$\frac{2x-1}{x+2} > -1$$
$$2x-1 > -1(x+2)$$
$$2x-1 > -x-2$$

$$3x > -1$$

$$x > -\frac{1}{3}$$

From process of deduction, we can say that the range of values will be from x → infinity, as one of the answers is 3, and the denominator is 3, so it will tend to infinity

Therefore subbing in

$$x = -\frac{1}{3}$$

$$= \frac{2(-\frac{1}{3})-1}{3-(-\frac{1}{3})}$$

$$= -\frac{1}{2}$$

Therefore

$$L > -\frac{1}{2}$$

65)

$$2\sum_{r=0}^{\infty}[\log_2 a]^r = \sum_{k=1}^{\infty}[1+b]^{-k} \quad \text{and} \quad 2\sum_{k=1}^{1}[1+b]^{-k} - \sum_{r=0}^{1}[\log_2 a]^r = \frac{7}{5}$$

Start by working on the LHS equation

We can see this will be a geometric series:

Using the formula:

$$sum\ to\ infintiy = \frac{a}{1-r}$$

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$a = 2$

$r = \log_2 a$

$$sum\ to\ infintiy = \frac{2}{1 - \log_2 a}$$

Now looking at the RHS of the same equation which is also a geometric series:

a = $(1-b)^{-1}$

r = $(1-b)^{-1}$

$$sum\ to\ infintiy = \frac{\frac{1}{1+b}}{1 - \frac{1}{1+b}}$$

$$= \frac{\frac{1}{1+b}}{\frac{1+b-1}{1+b}}$$

$$= \frac{\frac{1}{1+b}}{\frac{b}{1+b}}$$

$$= \frac{1}{b}$$

So equating the terms:

$$\frac{2}{1 - \log_2 a} = \frac{1}{b}$$

$$2b = 1 - \log_2 a \quad (1)$$

Now looking at the RHS equation

This is just the sum of a few terms, so we can write them out

For the first term:

$$= \frac{1}{1+b}$$

For the second term:

$$= 1 + \log_2 a$$

Putting the terms together

$$\frac{1}{1+b} - (1 - \log_2 a) = \frac{7}{5}$$

simplifying

$$\frac{1}{1+b} - \log_2 a = \frac{12}{5} \quad (2)$$

Using simultaneous equations on (1) and (2)

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$$2b = 1 - \log_2 a \quad 1$$

$$\frac{1}{1+b} - \log_2 a = \frac{12}{5} \quad 2$$

Writing this in a simpler form:

$$2b + \log_2 a = 1$$

$$\frac{1}{1+b} - \log_2 a = \frac{12}{5}$$

Using

$$\log_2 a = 1 - 2b$$

Subbing into 2

$$\frac{1}{1+b} - 1 + 2b = \frac{12}{5}$$

Solving for b

$$b = \frac{3}{2}, -\frac{4}{5}$$

Now using:

$$\log_2 a = 1 - 2\left(\frac{3}{2}\right)$$

$$\log_2 a = 1 - 2\left(-\frac{4}{5}\right)$$

$$a = \frac{1}{4}, 2^{\frac{13}{5}}$$

Therefore:

(a,b) = $\left(\frac{1}{4}, \frac{3}{2}\right)$= $\left(2^{\frac{13}{5}}, -\frac{4}{5}\right)$

66)

i.

Written out this looks like $u_{k+1} + u_{k+2} + u_{k+3} + \cdots$

The (k−1)th term = 108 and the kth term = 81

(k−1)th and kth terms are 2 consecutive terms, so we can use these to work out a common ratio

$$ratio = \frac{k}{k-1} = \frac{81}{108} = \frac{3}{4}$$

kth term = 81 means $u_k = 81$. We can use this to find a

$$ar^{k-1} = 81$$

$$a\left(\frac{3}{4}\right)^{k-1} = 81$$

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$$a\left(\frac{3}{4}\right)^k\left(\frac{3}{4}\right)^{-1} = 81$$

$$a\left(\frac{3}{4}\right)^k = \frac{81}{\frac{3}{4}}$$

This can help us find u_{k+1}

$u_{k+1} = ar^{k+1-1} = a\left(\frac{3}{4}\right)^k = \frac{81}{\frac{3}{4}} = 81 \times \frac{3}{4} = \frac{243}{4}$

u_{k+1} is the first term in the series hence $a = \frac{243}{4}$

We know $r = \frac{3}{4}$

Way 1:

we want

$$\sum_{n=k+1}^{\infty} u_n$$

so now we need to find the sum to infinity starting at the k+1 term

Using the sum to infinity formula

$$= \frac{a}{1-r}$$

a = $\frac{243}{4}$
$r = \frac{3}{4}$

$$= \frac{\frac{243}{4}}{1-\frac{3}{4}}$$

$$= 243$$

67)

This is a sum embedded into a sum, so work on each one individually

Starting with the inner sum:

Write out the first few terms:

$$= 2^1 + 2^2 + 2^3 \ldots 2^k$$

This is a sum of a geometric series, so we need the formula

$$sum = \frac{a(1-r^n)}{1-r}$$

The constants are:
a = 2

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r = 2
n = k

$$sum = \frac{2(1-2^k)}{1-2}$$
$$sum = \frac{2(1-2^k)}{-1}$$
$$sum = -2(1-2^k)$$

Now we have:

$$\sum_{k=1}^{n} -2 + 2^{k+1}$$

Splitting this up into 2 sums

$$\sum_{k=1}^{n} -2 + 2^{k+1} = \sum_{k=1}^{n} -2 + \sum_{k=1}^{n} 2^{k+1}$$

Looking at the blue = -2n

Using:

$$sum\ to\ infintiy = \frac{a(1-r^n)}{1-r}$$

a = 4
r = 2

$$sum\ to\ n = \frac{4(1-2^n)}{1-2}$$
$$sum\ = \frac{4(1-2^n)}{-1}$$
$$sum\ = -4(1-2^n)$$

Adding the two terms

$$= -4(1-2^n) - 2\text{n}$$

Expand and simplify

$$= 2^{n+2} - 2n - 4$$

68)

Splitting this up into 2 terms

$$\sum_{n=1}^{\infty} \frac{3^n - 2}{4^{n+1}} = \sum_{n=1}^{\infty} \frac{3^n}{4^{n+1}} + \sum_{n=1}^{\infty} \frac{-2}{4^{n+1}}$$

This is a sum of a geometric series, so we need the formula

$$sum = \frac{a}{1-r}$$

Writing out the first few terms:

$$\frac{3^1}{4^{1+1}} + \frac{3^2}{4^{2+1}} + \frac{3^3}{4^{3+1}}$$

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The constants are:
a = $\frac{3}{16}$
r = $\frac{3}{4}$

$$sum = \frac{\frac{3}{16}}{1-\frac{3}{4}}$$
$$sum = \frac{3}{4}$$

Now using the red equation:

Writing out the first few terms:

$$\frac{-2}{4^{1+1}} + \frac{-2}{4^{2+1}} + \frac{-2}{4^{3+1}}$$

The constants are:
A = -$\frac{1}{8}$
R = $\frac{1}{4}$

$$sum = \frac{-\frac{1}{8}}{1-\frac{1}{4}}$$
$$sum = -\frac{1}{6}$$

Adding the terms

$$\frac{3}{4} - \frac{1}{6} = \frac{7}{12}$$

69)

Start by looking at the LHS of the first equation

$$sum = \frac{a}{1-r}$$

The constants are:
A = 1
R = R

$$sum = \frac{1}{1-R}$$

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Now on the RHS

The constants are:
A = 1
R = r

$$sum = \frac{1}{1-r}$$

Writing it in the required form:

$$\frac{1}{1-R} = \left(\frac{1}{1-r}\right)^2$$

$$R = 2r - r^2$$

Using the second equation
The constants are:
A = 1
R = $\frac{r}{2R}$

$$sum = \frac{1}{1-\frac{r}{2R}}$$

Simplifying:

$$sum = \frac{2R}{2R-r}$$

Notice how none of the terms in the question has a R term:

$$R = 2r - r^2$$

$$sum = \frac{2(2r-r^2)}{2(2r-r^2)-r}$$

$$sum = \frac{4r-2r^2}{3r-2r^2}$$

Cancel out a r term

$$sum = \frac{4-2r}{3-2r}$$

From the numerator, take out a factor of 2

$$sum = \frac{2(2-r)}{3-2r}$$